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Applications of Mazurkiewicz type sets in the study of measurability properties of sets and functions

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Mazurkiewicz set

In 1914, S. Mazurkiewicz presented a transfinite constructions of a subset A of the Euclidian plane $\mathbf{R^2}$, having the extraordinary property.

Definition

A set $M \subset \mathbf{R}^2$ is called Mazurkiewicz subset of $\mathbf{R}^2$ if $card(M \cap l) = 2$ for every straight line in $\mathbf{R}^2$.

Auxiliary notions

Let $M(\mathbf{R}^2)$ be a class of all nonzero σ-finite translation invariant measures on $\mathbf{R}^2$.

Definition

A set $X \subset \mathbf{R}^2$ is called *negligible* with respect to $M(\mathbf{R}^2)$ if these two conditions are satisfied for X:

- there exists a measure $\nu \in M(\mathbf{R}^2)$ such that $X \in dom(\nu)$
- for any measure $\mu \in M(\mathbf{R}^2)$, the relation $X \in dom(\mu)$ implies the equality $\mu(X) = 0$

Auxiliary notions

Definition

A set $Y \subset \mathbf{R}^2$ is called *absolutely negligible* with respect to $M(\mathbf{R}^2)$ if, for every measure $\mu \in M(\mathbf{R}^2)$, there exists a measure $\mu' \in M(\mathbf{R}^2)$ such that the relations

$$\mu' extends \mu, Y \in dom(\mu'), \mu'(Y) = 0$$

hold true.

Let us remark that any $\mathbf{R}^2$-absolutely negligible set is also $\mathbf{R}^2$-negligible, but the converse assertion fails to valid.

Auxiliary Notions

Definition

A set $T \subset \mathbf{R}^2$is called almost $\mathbf{R}^2$-invariant if $card(T) = \mathbf{c}$ and

$$card((h + T) \triangle T) < \mathbf{c}$$

for each vector $h \in \mathbf{R}^2$

.

Definition

A se $W \subset \mathbf{R}^2$ is called λ_2-thick in $\mathbf{R}^2$ if $B \cap W \neq \emptyset$ for every Borel set $B \subset \mathbf{R}^2$ with $\lambda_2(B) > 0$.

Let e be an arbitrary nonzero vector in $\mathbf{R}^2$.

Definition

A set $A \subset \mathbf{R}^2$ is called

- *uinform* in direction e if $card(l \cap A) \leq 1$;

for any straight line $l \subset \mathbf{R}^2$ parallel to e.

.

The above definition immediately implies that, for any nonzero vector $e \in \mathbf{R}^2$, the Mazurkiewicz set is finite in direction e.

Auxiliary Notions

Let e be an arbitrary nonzero vector in $\mathbf{R}^2$.

Definition

A set $A \subset \mathbf{R}^2$ is called

- *uinform* in direction e if $card(l \cap A) \leq 1$;
- *finite* in direction e if $card(l \cap A) < \omega$;

for any straight line $l \subset \mathbf{R}^2$ parallel to e.

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The above definition immediately implies that, for any nonzero vector $e \in \mathbf{R}^2$, the Mazurkiewicz set is finite in direction e.

Auxiliary Notions

Let e be an arbitrary nonzero vector in $\mathbf{R}^2$.

Definition

A set $A \subset \mathbf{R}^2$ is called

- *uinform* in direction e if $card(l \cap A) \leq 1$;
- *finite* in direction e if $card(l \cap A) < \omega$;
- *countable* in direction e if $card(l \cap A) \leq \omega$;

for any straight line $l \subset \mathbf{R}^2$ parallel to e.

.

The above definition immediately implies that, for any nonzero vector $e \in \mathbf{R}^2$, the Mazurkiewicz set is finite in direction e.

Theorem

If $M \subset \mathbf{R}^2$ is finite in some direction l, then M is negligible with respect to the class $M(\mathbf{R}^2)$.

Corollary

Every Mazurkiewicz set is negligible with respect to the same class of measures.

Auxiliary proposition

Lemma

Every Hamel basis of the space $\mathbf{R}^n$ is absolutely negligible subset of $\mathbf{R}^n$.

Independent, in the papers

1. M. Bienias; S. Gab ; R. Rałowski; S. Zeberski *Two point sets with additional properties* Czechoslovak Mathematical Journal, Vol. 63 (2013), No. 4, 10191037
2. A. Kharazisvili, *On property of Hamel bases* (in Russian) Bull. Acad. Sci. GSSR, 95, 2 (1979)

was proved next Lemma:

Lemma

There exists a Mazurkiewic subset of $\mathbf{R}^2$ which is a Hamel basis of $\mathbf{R}^2$.

From above mentioned two Lemmas concludes:

Theorem

There exists a Mazurkiewicz subset X of $\mathbf{R}^2$ which is absolutely negligible with respect to the class $M(\mathbf{R}^2)$. So, for any measure $\mu \in M(\mathbf{R}^2)$, there exists a measure $\mu' \in M(\mathbf{R}^2)$, extending μ and such that $X \in dom(\mu')$ and $\mu'(X) = 0$

Auxiliary proposition

Lemma

Let e be a nonzero vector in $\mathbf{R}^2$ and let Z be a subset of $\mathbf{R}^2$ countable in direction e. Then there exists a set $Z_0 \subset \mathbf{R}^2$ and a countable family $\{h_n : n < \omega\} \subset \mathbf{R}^2$ such that:

1. Z_0 is uniform in direction e;
2. $Z_0 \subset \cup\{h_n + Z_0 : n < \omega\}$.

Auxiliary proposition

Lemma

Let G, A and B satisfy the following condition:

1. G is a subgroup of the additive group $(\mathbf{R}^2, +)$ and $card(G) < \mathbf{c}$
2. A is a subset of $\mathbf{R}^2$ and $card(A) < \mathbf{c}$;
3. B is a λ_2-measurable subset of $\mathbf{R}^2$ with $\lambda_2(B) > 0$.

Then there exists a point $z \in B$ such that:

- i $(G + z) \cap A = \emptyset$
- ii for any two distinct points $a \in A$ and $a' \in A$, the line $l(a, a')$ does not intersects the orbit $G + z$;
- iii for any two distinct points $x \in G + z$ and $y \in G + z$, the line $l(x, y)$ does not intersect the set A.

Auxiliary notions

Lemma

Let Γ be any countably infinite non-collinear subgroup of $\mathbf{R}^2$. There exists a Mazurkiewicz set Z such that $\Gamma + Z$ has the following property: for each countable family $\{h_m : m < \omega\} \subset \mathbf{R}^2$ the set

$$\cap\{h_m + \Gamma + Z : m < \omega\}$$

is λ_2-thick in $\mathbf{R}^2$ and the equality

$$card(\cap\{h_m + \Gamma + Z : m < \omega\}) = \mathbf{c}$$

holds true.

Main results

Theorem

Let Z and Γ be as in least Lemma. There exists a measure ν on $\mathbf{R}^2$ such that:

1. ν is an extension of λ_2;
2. ν is translation invariant
3. $(\Gamma + Z) \in dom(\nu)$ and $\nu(\mathbf{R}^2 \setminus (\Gamma + Z)) = 0$.

In particular, the Mazurkiewicz set Z is not $\mathbf{R}^2$-absolutely negligible in $\mathbf{R}^2$.

Conclusion

In general, we can characterize the Mazurkiewicz sets with respect $M(\mathbf{R}^2)$ class:

- all Mazurkiewicz sets are negligible with respect $M(\mathbf{R}^2)$;

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Conclusion

In general, we can characterize the Mazurkiewicz sets with respect $M(\mathbf{R}^2)$ class:

- all Mazurkiewicz sets are negligible with respect $M(\mathbf{R}^2)$;
- there exists a Mazurkiewicz set, which absolutely negligible with respect $M(\mathbf{R}^2)$;
- there exists a Mazurkiewicz set, which is not-absolutely negligible with respect $M(\mathbf{R}^2)$.

References

1. M. Bienias; S. Gab ; R. Rałowski; S. Zeberski *Two point sets with additional properties* Czechoslovak Mathematical Journal, Vol. 63 (2013), No. 4, 10191037
2. A. Kharazisvili, *On property of Hamel bases* (in Russian) Bull. Acad. Sci. GSSR, 95, 2 (1979)
3. A. Kharazisvili, *Measurability properties of Mazurkiewicz sets* Bulletin of TICMI, 20, 2, (2016)

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